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Quiver Chern-Simons theories and crystals

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Abstract: We consider $\mathcal{N}=2$ quiver Chern-Simons theories described by brane tilings, whose moduli spaces are toric Calabi-Yau 4-folds. Simple prescriptions to obtain toric data of the moduli space and a corresponding brane crystal from a brane tiling are proposed.

Keywords: Brane Dynamics in Gauge Theories, Chern-Simons Theories, Conformal Field Models in String Theory, M-Theory.

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Contents

1. Introduction

Recently, three-dimensional supersymmetric Chern-Simons theories have attracted great interest as theories for multiple M2-branes in various backgrounds. This was triggered by the proposal of $\mathcal{N} = 8$ interacting Chern-Simons theories by Bagger, Lambert [1−3], and Gusstavson [4, 5]. Their model (BLG model) is based on Lie 3-algebra, and the action includes structure constants, which satisfy the so-called fundamental identity. This model, however, has not succeeded in describing an arbitrary number of M2-branes in uncompactified flat background, due to the fact that the fundamental identity is very restrictive and it admits the only one non-trivial finite-dimensional algebra with a positive definite metric [6, 7]. The resulting theory is conjectured to describe two M2-branes on a certain orbifold [8, 9].

Aharony et al. proposed alternative model in [10], based on the recent progress in $\mathcal{N} = 4$ Chern-Simons theories [11, 12]. Their model (ABJM model) is $\mathrm{U}(N) \times \mathrm{U}(N)$ Chern-Simons gauge theory at level $(k, -k)$ with bi-fundamental matter fields. The model describes N M2-branes in the $\mathbf{C}^4/\mathbf{Z}_k$ orbifold background. Although only $\mathcal{N} = 6$ supersymmetry is manifest in the model, the supersymmetry is expected somehow to be enhanced to $\mathcal{N} = 8$ when $k = 1, 2$.

After the proposal of the ABJM model, some generalizations have been studied. Orbifolds of the ABJM model are discussed in [13−15]. In [14], a certain class of $\mathcal{N} = 3$ quiver Chern-Simons theories with non-toric moduli spaces are also studied based on the brane construction, and the hyper-Kähler toric structure of the moduli spaces is clarified in [16]. The moduli spaces of other superconformal Chern-Simons theories are studied in [17−21].

$\mathcal{N}=2$ Chern-Simons theories with general quiver structure are studied in [22], and it is shown how the gauge symmetries and D-term conditions are modified compared to the case of four-dimensional $\mathcal{N}=1$ gauge theories described by the same quiver diagrams. It is also found that the moduli spaces of such theories generically include a baryonic branch.

In the case of four-dimensional $\mathcal{N}=1$ supersymmetric gauge theories, brane tilings [23–25] are convenient tools to establish the relation between gauge theories and their moduli spaces, for a class of theories whose moduli spaces are toric Calabi-Yau 3-folds. See [26, 27] for review of brane tilings. Brane tilings are expected to be convenient for three-dimensional Chern-Simons theory, too. $\mathcal{N}=2$ quiver Chern-Simons theories described by brane tilings are studied in [28], and it is shown that the moduli space of the theories are toric Calabi-Yau 4-folds, and the Hilbert series is computed for some examples.

In this paper we consider the class of $\mathcal{N}=2$ quiver Chern-Simons theories described by brane tilings. Our aim is to establish the relation between brane tilings and brane crystals. Brane crystals are three-dimensional graphs proposed in [29–31] as diagrams describing three-dimensional superconformal field theories and the structure of their moduli spaces. We first give a simple prescription to obtain toric data of the moduli space from a tiling, and explain how we can construct a crystal describing the same moduli space.

This paper is organized as follows. In the next section, we explain the relation between brane tilings and quiver Chern-Simons theories. In section 3 we review how gauge symmetries of Chern-Simons theories are broken due to the existence of Chern-Simons terms following [22]. In section 4, we define gauge invariant operators which parameterize the moduli spaces of Chern-Simons theories. In section 5 we give a simple prescription to obtain the toric data of the moduli spaces by using tilings. This section has some overlap with [32]. The relation between brane tilings and brane crystals are discussed in 6. The last section is devoted to conclusions.

2. Tilings and Chern-Simons theories

We consider three-dimensional $\mathcal{N}=2$ quiver Chern-Simons theories described by brane tilings, which are also studied in [28].

A brane tiling is a bipartite graph drawn on $\mathbf{T}^2$. A bipartite graph is a graph consisting of vertices of two colors, say, white and black, and all links connect two vertices with different colors. Tilings have been used to describe four-dimensional $\mathcal{N}=1$ quiver gauge theories and the structure of their moduli spaces. The gauge group, the matter content, and the superpotential of a gauge theory can be read off from the brane tiling for the theory. Namely, faces correspond to $\mathrm{U}(N)$ factors in the gauge group, and links to bi-fundamental fields. The superpotential can be also read off from the tiling in the way we will mention later. These correspondences are naturally understood by regarding the tiling as a NS5-D5 system in type IIB string theory.

In this paper, we use tilings to describe three-dimensional $\mathcal{N}=2$ Chern-Simons theories. The gauge group, the matter content, and the superpotential are read off from the tiling in the same way as the four-dimensional case. These rules are naturally understood by regarding the tiling as a D4-NS5 system in type IIA theory, rather than the type IIB

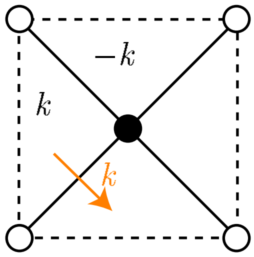

Figure 1: The tiling for the ABJM model at level $(k, -k)$. The arrow represents the flow $\boldsymbol{s}$ defined in (2.2).

brane system. By this reason, when we want to specify which of three or four dimensional theory a brane tiling describes, we call it IIB tiling (for four-dimensional theory), or IIA tiling (for Chern-Simons theory). Figure 1 shows an example of IIA tiling for the ABJM model.

Because we are here interested in the structure of the background spacetime probed by M2-branes, we discuss only abelian ($N = 1$) case. We use indices $I, J, \ldots$ for links, $i, j, \ldots$ for faces, and $a, b, \ldots$ for vertices. Let $\mathrm{U}(1)_i$ be the gauge group associated with face i, and Φ_I be the bi-fundamental field associated with link I. $I \in i$ means the link I is on the face i. The bi-fundamental field Φ_I is charged under two $\mathrm{U}(1)$ factors corresponding to the two faces sharing the link. The $\mathrm{U}(1)_i$ charge Q_{Ii} of the chiral multiplet Φ_I is uniquely determined by the bipartite graph. When the link I is not a side of the face i, $Q_{Ii} = 0$. Q_{Ii} is $+1$ (-1) if the left endpoint of the link I is black (white) when it is seen from the face i.

In order to specify a Chern-Simons theory, we need to fix the levels $k_i \in \mathbf{Z}$ for each gauge group as numbers assigned to faces in a IIA tiling. As is pointed out in [22], we need to impose the condition

$$\sum_i k_i = 0, \tag{2.1}$$

to obtain a four-dimensional moduli space. Because of this condition we can represent the levels k_i as

$$k_i = \sum_I Q_{Ii} s_I. \tag{2.2}$$

In order to interpret relations like (2.2) geometrically, we define two kinds of flows on the tiling. Let f_I be a set of numbers assigned to links. A normal flow $\boldsymbol{f}$ is a flow from faces to faces. We define the orientation of the flow to be the anti-clockwise direction around black vertices. ((a) in figure 2) The other flow associated with f_I is the tangential flow $\boldsymbol{f}^*$, which describes flow along links from black vertices to white ones. ((b) in figure 2) We can rewrite the relation (2.2) in terms of the normal flow $\boldsymbol{s}$ or the tangential flow $\boldsymbol{s}^*$ as

$$\{k_i\} = \operatorname{div} \boldsymbol{s} = \operatorname{rot} \boldsymbol{s}^*. \tag{2.3}$$

We will later see that this relation has natural interpretation in the context of brane realization of Chern-Simons theories.

For later convenience, we introduce the following normal flows. (See figure 3.)

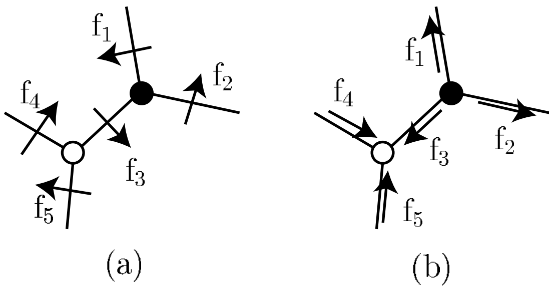

Figure 2: For a set of numbers f_I assigned to links we define two flows. (a) is a normal flow $\boldsymbol{f}$ and (b) is a tangential flow $\boldsymbol{f}^*$. These two flows are related by the $\pi/2$ rotation of arrows.

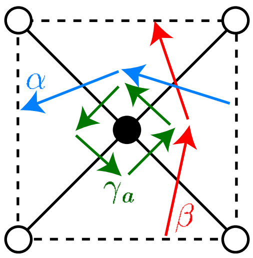

Figure 3: Examples of cycles $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}_a$ for the ABJM tiling are shown.

- $\boldsymbol{\alpha}$: a unit flow along α-cycle on the torus.
- $\boldsymbol{\beta}$: a unit flow along β-cycle on the torus.
- $\boldsymbol{\gamma}_a$: a unit flow around vertex a. The orientation is anti-clockwise (clockwise) around black (white) vertices.

An arbitrary conserved flow can be given as a linear combination of these flows. We define an operator $\mathcal{O}_{\boldsymbol{f}}$ for a normal flux $\boldsymbol{f}$ with non-negative integral components f_I by

$$\mathcal{O}_{\boldsymbol{f}} = \prod_I \Phi_I^{f_I}. \tag{2.4}$$

If $\boldsymbol{f}$ is conserved flow satisfying $\operatorname{div} \boldsymbol{f} = 0$, the operator is also defined for general N as a single or multiple trace operator, which is often called mesonic operators. Baryonic operators are associated with non-conserved flows. With this notation, the superpotential is represented as

$$W = \sum_a \pm \mathcal{O}_{\boldsymbol{\gamma}_a}, \tag{2.5}$$

where the signature of the summand is positive (negative) for black (white) vertices.

3. Gauge symmetries

In both three- and four-dimensional cases, the moduli space is defined as the coset X/G, where X is the manifold defined by the F-term conditions and G is the complexified gauge group. Because IIA and IIB tilings give the same F-term conditions, the manifold X is common to two cases. A difference arises in the gauge symmetry G. In this section we review how this difference arises following [22].

If the tiling has n faces, there are n U(1) factors. Among them, the diagonal U(1) decouples from matter fields, and the effective gauge symmetry is $\mathrm{U}(1)^{n-1}$. The complexification of this symmetry gives G in the IIB case. In the IIA case, however, it is known that $\mathrm{U}(1)^{n-1}$ is broken down to $\mathrm{U}(1)^{n-2}$ due to the existence of the Chern-Simons terms.

Let A_i be the $\mathrm{U}(1)_i$ gauge field. We define gauge fields

$$a = \sum_{i=1}^{n} A_i, \quad b = \sum_{i=1}^{n} k_i A_i. \tag{3.1}$$

Let c_k $(k = 1, \ldots, n-2)$ be linear comminations of A_i linearly independent of a and b. We can rewrite the Chern-Simons terms in the form

$$S_{\mathrm{CS}} = \frac{1}{2\pi n} \int b \wedge f + S'[b, c_k] \tag{3.2}$$

where $f = da$ and S' does not depend on the diagonal U(1) gauge field a. Because the gauge field a does not couple to matter fields, it appears only in the first term of (3.2). The action includes a only through f, and we can dualize it by introducing Lagrange multiplier τ and adding the following term to the action.

$$S_\tau = -\frac{1}{2\pi} \int d\tau \wedge f. \tag{3.3}$$

The equation of motion for f is

$$d\tau = \frac{1}{n} b. \tag{3.4}$$

Let us consider gauge transformation

$$A_i \to A_i + d\theta_i. \tag{3.5}$$

The relation (3.4) implies that the dual scalar field τ should be transformed under (3.5) by

$$\delta\tau = \frac{1}{n} \sum_{i=1}^{n} k_i \theta_i. \tag{3.6}$$

This non-linear gauge transformation of τ means that the gauge symmetry is always partially broken due to the vev of the scalar field τ. As is shown in [22] the period of τ is $2\pi/n$, and the parameters for unbroken gauge transformations should satisfy

$$2\pi\mathbf{Z} \ni \sum_{i=1}^{n} k_i \theta_i = \sum_{I,i} Q_{Ii} s_I \theta_i, \tag{3.7}$$

where we used (2.2) to obtain the final expression.

4. Gauge invariant operators

When we analyze the moduli space of a gauge theory, it is convenient to use gauge invariant operators as coordinates of the moduli space. In four-dimensional gauge theories described by IIB brane tiling, it is known that the moduli space is parameterized by three gauge invariant operators

$$\mathcal{M}_\alpha = \mathcal{O}_{\boldsymbol{\alpha}}, \quad \mathcal{M}_\beta = \mathcal{O}_{\boldsymbol{\beta}}, \quad \mathcal{W} = \mathcal{O}_{\boldsymbol{\gamma}_a}, \tag{4.1}$$

associated with the flows defined in section 2. For general N, these operators are defined as single-trace mesonic operators. $\mathcal{W}$ is one of the terms in the superpotential (2.5). Due to the F-term conditions, all terms in the superpotential have the same vev, and $\mathcal{W}$ as an element of the chiral ring does not depend on the choice of the vertex a. Because $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}_a$ generate arbitrary conserved flows, an arbitrary mesonic operator can be written as a function of these mesonic operators, and we can use these three as coordinates in the three-dimensional moduli space of four-dimensional gauge theory.

In the case of Chern-Simons theory, (4.1) are again gauge invariant operators, and we can use them as coordinates in the moduli space. However, we need another gauge invariant operator to parameterize the four-dimensional moduli space. Indeed, the restriction (3.7) of the gauge transformation parameters admits extra gauge invariant operators in addition to the mesonic operators in the four-dimensional gauge theory.

Let us consider an operator $\mathcal{O}_{\boldsymbol{q}}$ associated with a flow $\boldsymbol{q}$, which is not necessarily conserved. The field Φ_I associated with link I is transformed under the gauge transformation (3.5) as

$$\Phi_I \to \exp\left(i\sum_i Q_{Ii}\theta_i\right)\Phi_I, \tag{4.2}$$

and the gauge transformation of the operator $\mathcal{O}_{\boldsymbol{q}}$ is

$$\mathcal{O}_{\boldsymbol{q}} \to \exp\left(i\sum_{I,i} Q_{Ii}q_I\theta_i\right)\mathcal{O}_{\boldsymbol{q}}. \tag{4.3}$$

For the operator to be gauge invariant, the components q_I of the flow must satisfy

$$2\pi\mathbf{Z} \ni \sum_{I,i} Q_{Ii}q_I\theta_i. \tag{4.4}$$

If this condition were imposed for arbitrary θ_i, solutions would be given by $q_I = c_I$ where c_I is an arbitrary flow satisfying

$$\sum_I Q_{Ii}c_I = 0 \quad \forall\, i. \tag{4.5}$$

This is equivalent to $\operatorname{div}\boldsymbol{c} = 0$, and the normal flow $\boldsymbol{c}$ is conserved. Solutions in this form correspond to mesonic operators generated by (4.1).

The parameters θ_i are, however, constrained by (3.7) in the Chern-Simons theory. Thus we have an extra solution $q_I = s_I$, and a general solution is given by

$$q_I = ms_I + c_I, \tag{4.6}$$

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where m is an arbitrary integer and c_I is a conserved flow satisfying (4.5). Therefore, as the fourth coordinate on the moduli space, we should introduce the following baryonic operator associated with $\boldsymbol{s}$.

$$\mathcal{B} = \mathcal{O}[\boldsymbol{s}] = \prod_I \Phi_I^{s_I}. \tag{4.7}$$

An arbitrary gauge invariant operator in the Chern-Simons theory is given as a function of the four operators

$$\mathcal{M}_\alpha,\ \mathcal{M}_\beta,\ \mathcal{B},\ \mathcal{W}. \tag{4.8}$$

5. Toric data

In this section, we give a simple prescription to obtain toric data of the moduli space of Chern-Simons theory from the IIA brane tiling for the theory. The same subject is also investigated in [32].

In general a toric Calabi-Yau n-fold is represented as a $\mathbf{T}^n$ fibration over an n-dimensional polyhedral cone $\mathcal{C}$. The boundary of $\mathcal{C}$ consists of $(n-1)$-fans. On each $(n-1)$-fan a cycle v in the toric fiber shrinks. In other words, the fan is the fixed submanifold of the U(1) isometry generated by the vector v. For each $(n-1)$-fan, there is a vector v representing the shrinking cycle, and the toric data is given as a set of such vectors.

In order to extract the toric data of the moduli space from the information of a gauge theory, it is convenient to translate the system into a gauged linear sigma model (GLSM). This is achieved by solving the F-term conditions with the help of perfect matchings.

A perfect matching is a number assignment μ_I to links in a tiling which satisfies the following conditions.

- $\mu_I = 0$ or 1 for any link I.
- Among links ending on a vertex a, only one has non-vanishing f_I.

The following equation follows from these two conditions.

$$\langle \boldsymbol{\gamma}_a, \boldsymbol{\mu}^* \rangle \equiv \sum_{I \in a} f_I = 1 \quad \forall a, \tag{5.1}$$

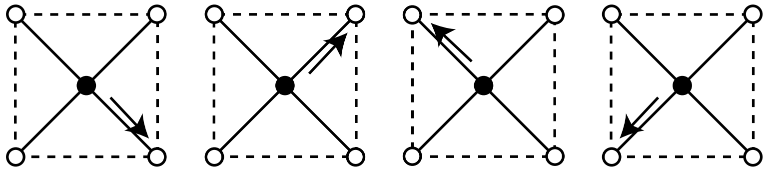
$\boldsymbol{\mu}^*$ is the tangential flow associated with the number assignment μ_I, and the product $\langle *, * \rangle$ is the intersection of a normal flow and a tangential flow, which is defined by

$$\langle \boldsymbol{f}, \boldsymbol{g}^* \rangle = \sum_I f_I g_I. \tag{5.2}$$

Figure 4 shows the four perfect matchings of the ABJM tiling.

The F-term conditions require all the terms $\mathcal{O}_{\boldsymbol{\gamma}_a}$ in the superpotential are the same. We can solve this condition by [33]

$$\Phi_I = \prod_{\mu \ni I} \rho_\mu, \tag{5.3}$$



$\boldsymbol{\mu}_1$ $\boldsymbol{\mu}_2$ $\boldsymbol{\mu}_3$ $\boldsymbol{\mu}_4$

Figure 4: The four perfect matchings of the ABJM tiling.

where ρ_μ is a GLSM field defined for each perfect matching μ, and the summation is taken over all the perfect matchings with $\mu_I = 1$. Substituting this into the definition of $\mathcal{O}_{\boldsymbol{\gamma}_a}$ and using (5.1), we can show that $\mathcal{O}_{\boldsymbol{\gamma}_a}$ is the product of all the GLSM fields regardless of the index a, and the F-term conditions are therefore satisfied by (5.3). Because this expression is redundant, we need to extend the gauge symmetry G acting on Φ_I by adding U(1) rotations of GLSM fields which keep Φ_I invariant. Let G' be this extended gauge symmetry. If the number of the perfect matchings is $n_{\rm pm}$ and the space spanned by the GLSM fields is $\mathbf{C}^{n_{\rm pm}}$, the moduli space of the gauge theory is the coset $\mathbf{C}^{n_{\rm pm}}/G'$.

In order to obtain the toric data, we need to find U(1) symmetries which have non-trivial fixed submanifolds. It is easy to show that in the moduli space defined as the coset $\mathbf{C}^{n_{\rm pm}}/G'$, such a submanifold is given as the image of the fixed plane $\rho_\mu = 0$ of $\mathrm{U}(1)_\mu$ symmetry by the homomorphism $\mathbf{C}^{n_{\rm pm}} \to \mathbf{C}^{n_{\rm pm}}/G'$, where $\mathrm{U}(1)_\mu$ is the symmetry which rotate only one GLSM field ρ_μ with charge 1. Thus, the components of the killing vector v_μ are given as the $\mathrm{U}(1)_\mu$ charges of the four toric coordinates.

As we mentioned above, we can use the four gauge invariant operators in (4.8) as coordinates in the Calabi-Yau 4-fold, and then the four components of v_μ are $\mathrm{U}(1)_\mu$ charges of these operators. By substituting (5.3) into (2.4) we rewrite an operator $\mathcal{O}_{\boldsymbol{q}}$ in terms of GLSM fields as

$$\mathcal{O}_{\boldsymbol{q}} = \prod_I \prod_{\mu \ni I} \rho_\mu^{q_I} = \prod_\mu \rho_\mu^{\langle \boldsymbol{q}, \boldsymbol{\mu}^* \rangle}, \tag{5.4}$$

and thus, the $\mathrm{U}(1)_\mu$ charge $[\mathcal{O}_{\boldsymbol{q}}]_\mu$ of the operator $\mathcal{O}_{\boldsymbol{q}}$ is given by

$$[\mathcal{O}_{\boldsymbol{q}}]_\mu = \langle \boldsymbol{q}, \boldsymbol{\mu}^* \rangle. \tag{5.5}$$

Applying this formula to the four gauge invariant operators, we obtain the following components of the killing vectors

$$v_\mu = \begin{pmatrix} [\mathcal{M}_\alpha]_\mu \\ [\mathcal{M}_\beta]_\mu \\ [\mathcal{B}]_\mu \\ [\mathcal{W}]_\mu \end{pmatrix} = \begin{pmatrix} \langle \boldsymbol{\alpha}, \boldsymbol{\mu}^* \rangle \\ \langle \boldsymbol{\beta}, \boldsymbol{\mu}^* \rangle \\ \langle \boldsymbol{s}, \boldsymbol{\mu}^* \rangle \\ 1 \end{pmatrix} \tag{5.6}$$

The last components of these vectors are always 1, and this guarantees that the toric manifold is Calabi-Yau. With this formula, we can easily obtain toric data from a given IIA tiling.

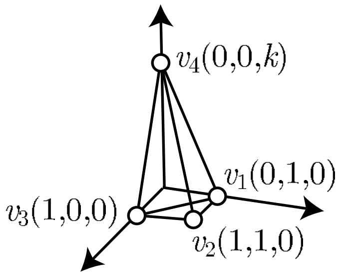


Figure 5: The toric diagram of the orbifold $\mathbf{C}^4/\mathbf{Z}_k$.

As a simple example, let us consider the ABJM tiling in figure 1. If we use the four perfect matchings in figure 4, and flows $\boldsymbol{s}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}_a$ in figure 1 and 3, we obtain the following four killing vectors.

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ k \\ 1 \end{pmatrix}. \tag{5.7}$$

By neglecting the fourth components of these vectors and plotting corresponding points in the three-dimensional lattice, we obtain the toric diagram of the moduli space $\mathbf{C}^4/\mathbf{Z}_k$. (figure 5)

6. Relation to crystals

A brane tiling describing a four-dimensional gauge theory can be regarded as a brane systems consisting of D5-branes and NS5-branes, which is T-dual to D3-branes probing a toric Calabi-Yau 3-fold, and the rules of reading off the gauge theory from the tiling have natural interpretation in terms of this brane system. For example, faces in a brane tiling represent D5-branes in the brane system, and the $\mathrm{U}(N)$ factors in the gauge group are identified with the gauge groups realized on the D5-branes.

Brane crystals [29−31] are analogues of brane tilings for M2-branes probing four-dimensional toric CY cones. By T-duality transformation in M-theory, a system of M2-branes probing a four-dimensional toric Calabi-Yau cone is transformed into a brane system consisting of M5-branes. Brane crystals are bipartite graphs in $\mathbf{T}^3$ representing the structure of the M5-brane systems [29].

Contrary to the case of brane tilings for four-dimensional gauge theories, we can obtain much less information from this brane system. This is because we have only little knowledge about theories realized on M5-brane systems. The purpose of this section is to obtain some information about the relation between Chern-Simons theories and brane crystals by using the results obtained in the previous sections.

The method to obtain toric data from crystals has been already known [30]. Actually, we may define brane crystals as bipartite graphs in $\mathbf{T}^3$ which give toric data of Calabi-Yau 4-folds in a similar way as brane tilings. The method to obtain the toric diagram from a

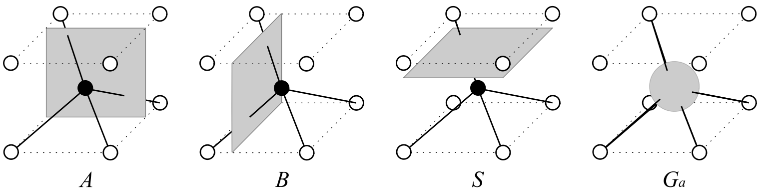


Figure 6: 2-cycles used to extract toric data from a crystal are shown. These are expected to represent M2-branes corresponding to gauge invariant operators [29, 30].

brane crystal is as follows: First, instead of the α and β cycles in brane tilings, we define three closed 2-cycles A (1-3 plane), B (2-3 plane), and S (1-2 plane). (See figure 6.) We assume these do not include vertices of the crystal on them. We also define a cycle G_a for each vertex a, which is a sphere enclosing the vertex a. (See figure 6.) Because the crystal is bipartite as well as tilings, we can define perfect matchings on it. If we denote the intersection number of a 2-cycle C and a perfect matching μ by $[C, \mu]$, the vectors v_μ forming the toric diagram are given by

$$v_\mu = \begin{pmatrix} [A, \mu] \\ [B, \mu] \\ [S, \mu] \\ [G_a, \mu] \end{pmatrix}. \tag{6.1}$$

By definition, the last component $[G_a, \mu]$ is always 1, and the Calabi-Yau condition is satisfied.

When a IIA tiling is given, it is easy to construct a brane crystal which gives the same toric data by the formula (6.1) as the data obtained by (5.6) from the IIA tiling. Let (x_a, y_a) be the coordinates of the vertex a in the IIA tiling. We put the corresponding vertex in the crystal at the point $(x_a, y_a, 0)$ in the three-dimensional torus. We make the three-dimensional graph by connecting these vertices in the same way as the tiling. Namely, if vertices a and b in the tiling are connected by a link, we connect the corresponding points in $\mathbf{T}^3$ by a link, too. There are infinitely many ways of connecting two vertices in $\mathbf{T}^3$ with a link with different winding numbers. We fix this ambiguity by requiring the following two conditions.

- The crystal reduces to the original tiling by the projection along the vertical axis.
- The vertical winding number of link I is s_I.

In other words, we interpret the integers s_I assigned to links as the gradient of links in the three-dimensional space. See figure 7 for an example of the ABJM model with $k = 2$. As the result, we obtain a three-dimensional bipartite graph with the same number of vertices and links as the original brane tiling (figure 7).

Let us confirm that the crystal constructed in this way correctly reproduces the toric data (5.6) obtained in the previous section. First of all, the two-dimensional and three-dimensional graphs are differ only by their embeddings to the tori. The former is embedded

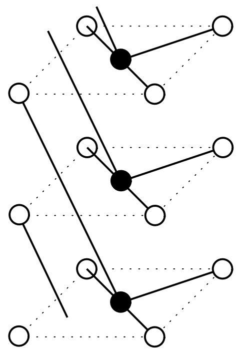


Figure 7: A crystal for ABJM model with $k = 2$ is shown. This figure includes two fundamental regions.

in $\mathbf{T}^2$ and the latter is in $\mathbf{T}^3$. Therefore, the three-dimensional graph has the same perfect matchings as the two-dimensional one.

Let us first consider the first two components $\langle \boldsymbol{\alpha}, \boldsymbol{\mu}^* \rangle$ and $\langle \boldsymbol{\beta}, \boldsymbol{\mu}^* \rangle$ in (5.6). We define the three-dimensional lift of α and β cycles by

$$A = \alpha \otimes \mathbf{S}^1, \quad B = \beta \otimes \mathbf{S}^1, \tag{6.2}$$

where $\mathbf{S}^1$ here is the cycle along the vertical direction. We use these 2-cycles as A and B in the formula (6.1). Then it is obvious that the first two components of (5.6) and those in (6.1) are the same.

$$\langle \boldsymbol{\alpha}, \boldsymbol{\mu}^* \rangle = [A, \mu], \quad \langle \boldsymbol{\beta}, \boldsymbol{\mu}^* \rangle = [B, \mu]. \tag{6.3}$$

For the last components, we define 2-cycle $\gamma_a \otimes \mathbf{S}^1$ for each vertex a. These are homologous to G_a defined above, and the relation

$$\langle \boldsymbol{\gamma}_a, \boldsymbol{\mu}^* \rangle = [G_a, \mu] \tag{6.4}$$

holds. (Actually, this is by definition always 1.)

Finally, let us consider the third component in (5.6), which is given as the $\mathrm{U}(1)_\mu$ charge of the baryonic operator $\mathcal{B}$. In the brane tiling, the baryonic operator is expressed in different way from the other mesonic operators. Mesonic operators are associated with conserved flows on the tiling, while the flow $\boldsymbol{s}$ corresponding to the baryonic operator $\mathcal{B}$ is not conserved. In the crystal, however, the third component is also given as the intersection of closed 2-cycle and perfect matchings. As we mentioned above, the link I in the crystal has the vertical winding s_I, and it intersects s_I times with the 2-cycle S. Therefore, the intersection $\langle \boldsymbol{s}, \boldsymbol{\mu}^* \rangle$ can be rewritten as the intersection of the closed 2-cycle S and the perfect matching μ.

$$\langle \boldsymbol{s}, \boldsymbol{\mu}^* \rangle = [S, \mu]. \tag{6.5}$$

Thus, the third components of (5.6) and (6.1) are the same.

Now we have confirmed that the crystal constructed above correctly reproduces the toric data of the moduli space of the Chern-Simons theory described by the tiling. At the same time, we have established the correspondence between the gauge invariant operators in (4.8) and closed 2-cycles in the crystal. Because the set of operators in (4.1) and (4.7) generates arbitrary gauge invariant operators, we have established the complete map between gauge invariant operators including both mesonic and baryonic ones and closed 2-cycle in the crystal, which are interpreted as closed M2-branes [29, 30]. An interesting feature of this correspondence is that even though in the Chern-Simons theory baryonic operators and mesonic operators have different structure, the brane crystal describes these in the parallel way.

As another support to our prescription, we can show that the level k_I are naturally obtained from the brane system described by the crystal. In order to read off the Chern-Simons theory from a crystal, we need to project the crystal along the vertical direction, and go back to the tiling. From the viewpoint of brane system, we can interpret this projection as the compactification of M-theory to type IIA string theory. Then, links and faces in the tiling are interpreted as a network of NS5-brane and D4-branes ending on the NS5-brane, respectively. Gauge groups are realized on the D4-branes, and the Chern-Simons terms are induced from the following boundary term in the D4-brane action.

$$S = \frac{1}{4\pi}\int_{\partial \mathrm{D4}} A \wedge dA \wedge d\phi, \tag{6.6}$$

where A is the gauge field on the D4-brane and ϕ is the compact scalar field on the NS5-brane corresponding to the X^{11} coordinate in the M-theory picture. If the scalar field ϕ has non-trivial profile along the boundary of the D4-brane, this induces the Chern-Simons coupling in the three-dimensional gauge theory, and the level is given by

$$k_i = \oint d\phi, \tag{6.7}$$

where the integration is taken over the boundary of the face i. If we identify $d\phi$ as the gradient s_I along links, (6.7) is nothing but the relation (2.2), or, equivalently, (2.3).

7. Conclusions

In this paper we investigated the relation between brane tilings describing $\mathcal{N} = 2$ Chern-Simons theories and the toric data of their moduli spaces. We gave a simple procedure to read off the toric data of the moduli space from the brane tiling. In order to obtain the toric data, we should first represent the Chern-Simons levels as a flow $\boldsymbol{s}$ on the tiling, and the vectors v_μ forming the toric diagram are obtained as the intersection of the perfect matchings $\boldsymbol{\mu}^*$ and the flows $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{s})$.

IIA brane tilings, which are regarded as brane systems consisting of D4-branes and NS5-branes, can be regarded as the projection of the crystals, which describe M5-brane systems. We showed that we can lift a IIA tiling to the corresponding crystal by using the flow $\boldsymbol{s}^*$ as the gradient of links. We found that gauge invariant operators, which include both mesonic and baryonic ones, are represented in the crystal as closed 2-cycles.

We emphasize that although our prescription always gives a crystal for a given IIA tiling, it is not always possible to give a tiling which reproduce a given crystal. Our

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prescription does not guarantee the existence of a Chern-Simons theory which reproduces a given toric Calabi-Yau 4-fold as its moduli space. There may not exist corresponding Chern-Simons theories for a class of manifolds. Contrary, there are crystals which gives more than two tilings by the projection along different directions. This may suggest the duality among Chern-Simons theories. We wish to come back to these issues in near future.

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